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ScienceDirect
Journal of Geometry and Physics 57 (2007) 1391–1403
JOURNAL OF GEOMETRY AND PHYSICS
www.elsevier.com/locate/jgp

Some deformations of nilpotent Lie superalgebras

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Received 27 July 2005; received in revised form 7 August 2006; accepted 3 November 2006
Available online 13 December 2006

Abstract

In this paper we study the infinitesimal deformations of the Lie superalgebra $L^{n,m}$. By means of these deformations all filiform Lie superalgebras can be obtained. In particular, we give a method that will allow us to determine the dimension of the space of deformations of type $\mathrm{Hom}(S^2(L_1^{n,m}), L_0^{n,m})$. Note that this type of deformation is the only one that occurs for Lie superalgebras which are not Lie algebras. Furthermore we develop a method for calculating a basis of the aforementioned space of deformations $\mathrm{Hom}(S^2(L_1^{n,m}), L_0^{n,m})$, giving it explicitly for $n \geq 2m - 1$.

MSC: 17B30; 17B56

Keywords: Lie superalgebras; Cohomology; Deformation; Nilpotent; Filiform

1. Introduction

In 1970 Vergne [12] used deformations to study the variety of finite-dimensional Lie algebras. She gave the definition of *filiform Lie algebras*, a class of nilpotent Lie algebras with important properties. In particular every filiform Lie algebra can be obtained by a deformation of the model filiform algebra L_n.

Concerning nilpotent Lie superalgebras, little work has been done. For example, we can find classifications in five dimensions [6,7], as well as a discussion of immediate problems appearing in this variety [4], and an approach to deformations of Lie superalgebras [5].

The present work is about nilpotent Lie superalgebras, in particular about the subclass of the so-called *filiform Lie superalgebras*. In the same way as filiform Lie algebras, all filiform Lie superalgebras can be obtained by infinitesimal deformations of the model Lie superalgebra $L^{n,m}$, thus being the analogue of the filiform Lie algebra L_n in the theory of Lie superalgebras.

We shall therefore consider infinitesimal deformations of $L^{n,m}$ which are defined by even 2-cocycles in $Z_0^2(L^{n,m}, L^{n,m})$. This latter space has an obvious direct decomposition into three subspaces according to the even

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doi:10.1016/j.geomphys.2006.11.001

part $L_0^{n,m}$ and the odd part $L_1^{n,m}$ of $L^{n,m}$. The first two components (which lie in $\mathrm{Hom}(\Lambda^2 L_0^{n,m}, L_0^{n,m})$ or in $\mathrm{Hom}(L_0^{n,m} \otimes L_1^{n,m}, L_0^{n,m})$ have already been dealt with in [9, Thm.3, Thm.4]. The aim of this article is a detailed study of the third subspace $Z^2(L^{n,m}, L^{n,m}) \cap \mathrm{Hom}(S^2 L_1^{n,m}, L_0^{n,m})$ which consists of 'symmetric' infinitesimal deformations, i.e. which are the only ones that give 'true' Lie superalgebras (i.e. which are not Lie algebras).
Moreover, representation theory of $\mathfrak{sl}(2,\mathbb{C})$ will allow us to determine the dimension of this space of deformations (Theorem 1); see also [1, p.197] for a similar $\mathfrak{sl}(2,\mathbb{C})$-type computation of the dimension of the scalar cohomology of the filiform Lie algebras.
Furthermore, we shall develop a method for calculating an expression for the basis of the above space of deformations, giving explicitly a basis for the case with more cocycles, that is $\frac{m(m+1)}{2}$ with $n \geq 2m-1$ (Theorem 2).

Combining with Khakimdjanov's results [9, Thm.3, Thm.4] we therefore obtain a *complete classification* of all the deformations of the Lie superalgebra $L^{n,m}$.

All the vector spaces that appear in this paper (and thus, all the algebras) are assumed to be $\mathbb{C}$-vector spaces of finite dimension. Moreover, we shall use the well-known convention that for the definition of a Lie (super-)bracket in terms of a basis only the nonvanishing brackets in some ordering of the base are explicitly mentioned.

2. Preliminaries

Recall that a *superspace* is a vector space with a $\mathbb{Z}_2$-grading: $V = V_0 \oplus V_1$. Usually, elements of the space V_0 are called even, and elements of the space V_1, odd; the indices 0 and 1 are modulo 2. A linear map $\phi : V \to W$ between two vector superspaces is called *even* iff $\phi(V_0) \subset W_0$ and $\phi(V_1) \subset W_1$ and is called *odd* iff $\phi(V_0) \subset W_1$ and $\phi(V_1) \subset W_0$. Clearly, $\mathrm{Hom}(V, W) = \mathrm{Hom}(V, W)_0 \oplus \mathrm{Hom}(V, W)_1$ where the first summand comprises all the even and the second summand all the odd linear maps. Tensor products $V \otimes W$ are $\mathbb{Z}_2$ graded by means of $(V \otimes W)_0 := (V_0 \otimes W_0) \oplus (V_1 \otimes W_1)$ and $(V \otimes W)_1 := (V_0 \otimes W_1) \oplus (V_1 \otimes W_0)$.

A *Lie superalgebra* (see [3,10]) is a superspace $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, with an even bilinear commutation operation (or "supercommutation") [,], which satisfies the conditions:

1. $[X, Y] = -(-1)^{\alpha\cdot\beta}[Y, X]\ \forall X \in \mathfrak{g}_\alpha, \forall Y \in \mathfrak{g}_\beta$.
2. $(-1)^{\gamma\cdot\alpha}[X, [Y, Z]] + (-1)^{\alpha\cdot\beta}[Y, [Z, X]] + (-1)^{\beta\cdot\gamma}[Z, [X, Y]] = 0$
   for all $X \in \mathfrak{g}_\alpha$, $Y \in \mathfrak{g}_\beta$, $Z \in \mathfrak{g}_\gamma$ with $\alpha, \beta, \gamma \in \mathbb{Z}_2$. (The graded Jacobi identity.)

Thus, $\mathfrak{g}_0$ is an ordinary Lie algebra, and $\mathfrak{g}_1$ is a module over $\mathfrak{g}_0$; the Lie superalgebra structure also contains the symmetric pairing $S^2\mathfrak{g}_1 \longrightarrow \mathfrak{g}_0$, which is a $\mathfrak{g}_0$-homomorphism and satisfies the graded Jacobi identity applied to three elements of the space $\mathfrak{g}_1$.

The *descending central sequence* of a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is defined by $\mathcal{C}^0(\mathfrak{g}) = \mathfrak{g}$, $\mathcal{C}^{k+1}(\mathfrak{g}) = [\mathcal{C}^k(\mathfrak{g}), \mathfrak{g}]$ for all $k \geq 0$. If $\mathcal{C}^k(\mathfrak{g}) = \{0\}$ for some k, the Lie superalgebra is called *nilpotent*. The smallest integer k such as $\mathcal{C}^k(\mathfrak{g}) = \{0\}$ is called the *nilindex* of $\mathfrak{g}$.

We define two new *descending sequences*, $\mathcal{C}^k(\mathfrak{g}_0)$ and $\mathcal{C}^k(\mathfrak{g}_1)$, as follows: $\mathcal{C}^0(\mathfrak{g}_i) = \mathfrak{g}_i$, $\mathcal{C}^{k+1}(\mathfrak{g}_i) = [\mathfrak{g}_0, \mathcal{C}^k(\mathfrak{g}_i)]$, $k \geq 0$, $i \in \{0, 1\}$.

If $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a nilpotent Lie superalgebra, then $\mathfrak{g}$ has super-nilindex or *s-nilindex* (p, q), if the following conditions hold:

$$(\mathcal{C}^{p-1}(\mathfrak{g}_0)) \neq 0 \qquad (\mathcal{C}^{q-1}(\mathfrak{g}_1)) \neq 0, \qquad \mathcal{C}^p(\mathfrak{g}_0) = \mathcal{C}^q(\mathfrak{g}_1) = 0.$$

Recall that a *module* $A = A_0 \oplus A_1$ of the Lie superalgebra $\mathfrak{g}$ is an even bilinear map $\mathfrak{g} \times A \to A$ satisfying

$$\forall X \in \mathfrak{g}_\alpha, \quad Y \in \mathfrak{g}_\beta a \in A : X(Ya) - (-1)^{\alpha\beta} Y(Xa) = [X, Y]a.$$

Lie superalgebra cohomology is defined in the following well-known way (see e.g. [3,11]): the superspace of q-*dimensional cocycles* of the Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with coefficients in the $\mathfrak{g}$-module $A = A_0 \oplus A_1$ is given by

$$C^q(\mathfrak{g}; A) = \bigoplus_{q_0+q_1=q} \mathrm{Hom}\left(\wedge^{q_0}\mathfrak{g}_0 \otimes S^{q_1}\mathfrak{g}_1, A\right).$$

This space is graded by $C^q(\mathfrak{g}; A) = C_0^q(\mathfrak{g}; A) \oplus C_1^q(\mathfrak{g}; A)$ with

$$C^q_p(\mathfrak{g}; A) = \bigoplus_{\substack{cq_0+q_1=q \\ q_1+r\equiv p \mod 2}} \mathrm{Hom}\left(\wedge^{q_0}\mathfrak{g}_0 \otimes S^{q_1}\mathfrak{g}_1, A_r\right).$$

The differential $d : C^q(\mathfrak{g}; A) \longrightarrow C^{q+1}(\mathfrak{g}; A)$ is defined by the formula

$$\begin{aligned}(dc)(g_1,\ldots,g_{q_0},h_1,\ldots,h_{q_1}) &= \sum_{1\le s<t\le q_0} (-1)^{s+t-1} c([g_s,g_t],g_1,\ldots,\hat{g}_s,\ldots,\hat{g}_t,\ldots,g_{q_0},h_1,\ldots,h_{q_1}) \\ &+ \sum_{s=1}^{q_0}\sum_{t=1}^{q_1}(-1)^{s-1} c(g_1,\ldots,\hat{g}_s,\ldots,g_{q_0},[g_s,h_t],h_1,\ldots,\hat{h}_t,\ldots,h_{q_1}) \\ &+ \sum_{1\le s<t\le q_1} c([h_s,h_t],g_1,\ldots,g_{q_0},h_1,\ldots,\hat{h}_s,\ldots,\hat{h}_t,\ldots,h_{q_1}) \\ &+ \sum_{s=1}^{q_0}(-1)^s g_s(c(g_1,\ldots,\hat{g}_s,\ldots,g_{q_0},h_1,\ldots,h_{q_1})) \\ &+ (-1)^{q_0-1}\sum_{s=1}^{q_1} h_s(c(g_1,\ldots,g_{q_0},h_1,\ldots,\hat{h}_s,\ldots,h_{q_1}))\end{aligned}$$

where $c \in C^q(\mathfrak{g}; A)$, $g_1,\ldots,g_{q_0} \in \mathfrak{g}_0$ and $h_1,\ldots,h_{q_1} \in \mathfrak{g}_1$. Obviously, $d\circ d = 0$, and $d(C^q_p(\mathfrak{g}; A)) \subset C^{q+1}_p(\mathfrak{g}; A)$ for $q = 0, 1, 2, \ldots$ and $p = 0, 1$. Then we have the *cohomology groups*

$$H^q_p(\mathfrak{g}; A) = Z^q_p(\mathfrak{g}; A)/B^q_p(\mathfrak{g}; A)$$

where the elements of $Z^q_0(\mathfrak{g}; A)$ and $Z^q_1(\mathfrak{g}; A)$ are called *even q-cocycles* and *odd q-cocycles* respectively. Analogously, the elements of $B^q_0(\mathfrak{g}; A)$ and $B^q_1(\mathfrak{g}; A)$ will be *even q-coboundaries* and *odd q-coboundaries* respectively. Two elements of $Z^q(\mathfrak{g}; A)$ are said to be *cohomologous* if their residue classes modulo $B^q(\mathfrak{g}; A)$ coincide, i.e., if their difference lies in $B^q(\mathfrak{g}; A)$.

3. Deformations of $L^{n,m}$

If we denote by $\mathcal{N}^{n+1,m}$ the variety of nilpotent Lie superalgebras $\mathfrak{g} = \mathfrak{g}_0\oplus\mathfrak{g}_1$ with $\dim\mathfrak{g}_0 = n+1$ and $\dim\mathfrak{g}_1 = m$ we will have the following definition:

Definition 3.1 ([4]). Any nilpotent Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \in \mathcal{N}^{n+1,m}$ with s-nilindex (n, m) is called *filiform*.

We denote by $\mathcal{F}^{n+1,m}$ the subset of $\mathcal{N}^{n+1,m}$ consisting of all the filiform Lie superalgebras.

Before we study this family of Lie superalgebras it is convenient to solve the problem of finding a suitable basis, a so-called *adapted basis*.

Theorem 3.1.1 ([4]). *If $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \in \mathcal{F}^{n+1,m}$, then there exists an adapted basis of $\mathfrak{g}$, namely $\{X_0, X_1,\ldots,X_n, Y_1,\ldots,Y_m\}$, with $\{X_0, X_1,\ldots,X_n\}$ a basis of $\mathfrak{g}_0$ and $\{Y_1,\ldots,Y_m\}$ a basis of $\mathfrak{g}_1$, such that:*

$$\begin{cases}[X_0, X_i] = X_{i+1}, & 1\le i\le n-1,\\ [X_0, X_n] = 0, & \\ [X_0, Y_j] = Y_{j+1}, & 1\le j\le m-1,\\ [X_0, Y_m] = 0. & \end{cases}$$

X_0 is called the characteristic vector.

From the preceding theorem it can be observed that the simplest filiform Lie superalgebra, denoted by $L^{n,m}$, will be defined by the following brackets:

$$L^{n,m} : \begin{cases}[X_0, X_i] = X_{i+1}, & 1\le i\le n-1,\\ [X_0, Y_j] = Y_{j+1}, & 1\le j\le m-1,\end{cases}$$

with $\{X_0, X_1,\ldots,X_n, Y_1,\ldots,Y_m\}$ a basis of $L^{n,m}$. We shall frequently write

$$V_0 := \langle X_1,\ldots,X_n\rangle, \qquad V_1 := \langle Y_1,\ldots,Y_m\rangle,$$

whence $L_0^{n,m} = \langle X_0\rangle \oplus V_0$ and $L_1^{n,m} = V_1$.

This superalgebra will be the most important filiform Lie superalgebra since all the other filiform Lie superalgebras can be obtained from $L^{n,m}$ by deformations (in complete analogy to Lie algebras; see [12]). So, we are going to consider its infinitesimal deformations that will be given by the even 2-cocycles, $Z_0^2(L^{n,m}, L^{n,m})$.

An infinitesimal deformation of $L^{n,m}$ will thus be an element of the following space:

$$\begin{aligned} Z_0^2(L^{n,m}, L^{n,m}) = \;& Z^2(L^{n,m}, L^{n,m}) \cap \mathrm{Hom}(\mathfrak{g}_0 \wedge \mathfrak{g}_0, \mathfrak{g}_0) \oplus Z^2(L^{n,m}, L^{n,m}) \cap \mathrm{Hom}(\mathfrak{g}_0 \otimes \mathfrak{g}_1, \mathfrak{g}_1)\\ &\oplus Z^2(L^{n,m}, L^{n,m}) \cap \mathrm{Hom}(S^2\mathfrak{g}_1, \mathfrak{g}_0)\\ =:\;& A \oplus B \oplus C. \end{aligned}$$

We shall frequently simplify the notation and write $\mathfrak{g}_0$ and $\mathfrak{g}_1$ in place of $L_0^{n,m}$ and $L_1^{n,m}$.

The subspaces $A = Z^2(L^{n,m}, L^{n,m}) \cap \mathrm{Hom}(\mathfrak{g}_0 \wedge \mathfrak{g}_0, \mathfrak{g}_0)$ and $B = Z^2(L^{n,m}, L^{n,m}) \cap \mathrm{Hom}(\mathfrak{g}_0 \otimes \mathfrak{g}_1, \mathfrak{g}_1)$ have been completely determined by Khakimdjanov; see [9]. Our aim is therefore to determine the remaining subspace C:

$$C = Z^2(L^{n,m}, L^{n,m}) \cap \mathrm{Hom}(S^2\mathfrak{g}_1, \mathfrak{g}_0).$$

The importance of C can be summed up in three facts:

1. The deformations that belong to $A \oplus B$ lead to Lie superalgebras that always are Lie algebras (split cases, since the odd part remains an abelian Lie subalgebra). However, the deformations of $L^{n,m}$ that belong to C lead to genuine Lie superalgebras (that they are not Lie algebras).
2. The infinitesimal deformations belonging to C are all integrable.
3. The nonzero infinitesimal deformations of C are never cohomologous to 0, that is $C \cap B_0^2(L^{n,m}, L^{n,m}) = 0$.

4. $\mathfrak{sl}(2, \mathbb{C})$-module method

In this section we are going to explain the $\mathfrak{sl}(2, \mathbb{C})$-module method to compute the dimension of C. Recall the following well-known facts about the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ and its finite-dimensional modules; see e.g. [2,8]:

$\mathfrak{sl}(2, \mathbb{C}) = \langle X_-, H, X_+\rangle$ with the following commutation relations:

$$\begin{cases} [X_+, X_-] = H\\ [H, X_+] = 2X_+,\\ [H, X_-] = -2X_-. \end{cases}$$

Let V be a n-dimensional $\mathfrak{sl}(2, \mathbb{C})$-module, $V = \langle e_1, \dots, e_n\rangle$. Then, up to isomorphism there exists a unique structure of an irreducible $\mathfrak{sl}(2, \mathbb{C})$-module in V given in a basis $e_1, \dots, e_n$ as follows [2]:

$$\begin{cases} X_+ \cdot e_i = e_{i+1}, & 1 \le i \le n-1,\\ X_+ \cdot e_n = 0, & \\ H \cdot e_i = (-n + 2i - 1)e_i, & 1 \le i \le n. \end{cases}$$

It is easy to see that e_n is the maximal vector of V and its weight, called the highest weight of V, is equal to $n - 1$.

Let $V_0, V_1, \dots, V_k$ be $\mathfrak{sl}(2, \mathbb{C})$-modules, then the space $\mathrm{Hom}(\otimes_{i=1}^k V_i, V_0)$ is a $\mathfrak{sl}(2, \mathbb{C})$-module in the following natural manner:

$$(\xi \cdot \varphi)(x_1, \dots, x_k) = \xi \cdot \varphi(x_1, \dots, x_k) - \sum_{i=1}^{i=k} \varphi(x_1, \dots, \xi \cdot x_i, x_{i+1}, \dots, x_n)$$

with $\xi \in \mathfrak{sl}(2, \mathbb{C})$ and $\varphi \in \mathrm{Hom}(\otimes_{i=1}^k V_i, V_0)$. An element $\varphi \in \mathrm{Hom}(V_1 \otimes V_1, V_0)$ is said to be invariant if $X_+ \cdot \varphi = 0$, that is

$$X_+ \cdot \varphi(x_1, x_2) - \varphi(X_+ \cdot x_1, x_2) - \varphi(x_1, X_+ \cdot x_2) = 0, \quad \forall x_1, x_2 \in V_1. \tag{4.1}$$

Note that $\varphi \in \mathrm{Hom}(V_1 \otimes V_1, V_0)$ is invariant if and only if φ is a maximal vector.

On the other hand, we are going to consider the Lie superalgebra $L^{n,m}$ with basis $\{X_0, X_1, \dots, X_n, Y_1, \dots, Y_m\}$. By definition, a cocycle φ belonging to C will be a symmetric bilinear map:

$$\varphi : S^2 L_1^{n,m} \longrightarrow L_0^{n,m}$$

such that $d\varphi = 0$. That is, φ will satisfy the two conditions

$$[x, \varphi(y, z)] + [z, \varphi(x, y)] + [y, \varphi(z, x)] = 0, \quad \forall x, y, z \in L_1^{n,m}$$
$$[x, \varphi(y, z)] - \varphi(z, [x, y]) - \varphi(y, [x, z]) = 0, \quad \forall x \in L_0^{n,m}; y, z \in L_1^{n,m}.$$

Taking into account the law of $L^{n,m}$ the above conditions can be simplified to

$$[X_0, \varphi(Y_i, Y_j)] - \varphi([X_0, Y_i], Y_j) - \varphi(Y_i, [X_0, Y_j]) = 0, \quad \text{with } 1 \le i \le j \le m. \tag{4.2}$$

We are going to consider the structure of irreducible $\mathfrak{sl}(2, \mathbb{C})$-module in $V_0 = \langle X_1, \ldots, X_n \rangle = L_0^{n,m}/\mathbb{C}X_0$ and in $V_1 = \langle Y_1, \ldots, Y_n \rangle = L_1^{n,m}$; thus in particular:

$$\begin{cases} X_+ \cdot X_i = X_{i+1}, & 1 \le i \le n-1, \\ X_+ \cdot X_n = 0, & \\ X_+ \cdot Y_j = Y_{j+1}, & 1 \le j \le m-1, \\ X_+ \cdot Y_m = 0. & \end{cases}$$

We identify the multiplication of X_+ and X_i in the $\mathfrak{sl}(2, \mathbb{C})$-module $V_0 = \langle X_1, \ldots, X_n \rangle$ with the bracket $[X_0, X_i]$ in $L_0^{n,m}$. Analogously, we identify $X_+ \cdot Y_j$ and $[X_0, Y_j]$. Thanks to these identifications, the expressions (4.1) and (4.2) are equivalent, so we have the following result:

Proposition 4.1. *Any symmetric bilinear map $\varphi, \varphi : S^2 V_1 \longrightarrow L_0^{n,m}$ will be an element of C if and only if φ is a maximal vector of the $\mathfrak{sl}(2, \mathbb{C})$-module* $\mathrm{Hom}(S^2 V_1, V_0)$*, with $V_0 = \langle X_1, \ldots, X_n \rangle$ and $V_1 = L_1^{n,m}$.*

Proof. If $\varphi \in C$, then $\mathrm{Im}\,\varphi \subset V_0$. In fact, the ideal V_0 is equal to its own centralizer in $L_0^{n,m}$. Let $x \in V_0$ and $y, y' \in L_1^{n,m}$. Thanks to the cocycle identity we get

$$[x, \varphi(y, y')] = \varphi(y, [x, y']) + \varphi(y', [x, y]) = 0 \quad \forall x \in V_0; y, y' \in L_1^{n,m}.$$

since $[V_0, V_1] = \{0\}$. Hence $\varphi(y, y')$ centralizes V_0, and is thus an element of V_0. The same cocycle equation for x replaced by X_0, i.e.

$$[X_0, \varphi(y, y')] - \varphi(y, [X_0, y']) - \varphi(y', [X_0, y]) = 0,$$

shows that $\varphi \in \mathrm{Hom}(S^2 V_1, V_0)$ is invariant which proves the proposition and the Eq. (4.2). □

Corollary 4.1.1. *As each irreducible $\mathfrak{sl}(2, \mathbb{C})$-module has (up to nonzero scalar multiples) a unique maximal vector, then the dimension of C is equal to the number of summands of any decomposition of* $\mathrm{Hom}(S^2 V_1, V_0)$ *into the direct sum of irreducible $\mathfrak{sl}(2, \mathbb{C})$-modules.*

But instead of looking at the maximal vectors, we can equally well use the fact that each irreducible module contains either a unique (up to scalar multiples) vector of weight 0 (in the case where the dimension of the irreducible module is odd) or a unique (up to scalar multiples) vector of weight 1 (in the case where the dimension of the irreducible module is even). We therefore have the

Corollary 4.1.2. *The dimension of C is equal to the dimension of the subspace of* $\mathrm{Hom}\,(S^2 V_1, V_0)$ *spanned by the vectors of weight* 0 *or* 1.

5. Computation of the dimension of $Z^2(L^{n,m}, L^{n,m}) \cap \mathrm{Hom}(S^2\mathfrak{g}_1, \mathfrak{g}_0)$

In this section we are going to apply the $\mathfrak{sl}(2, \mathbb{C})$-module method in order to obtain the dimension of $C = Z^2(L^{n,m}, L^{n,m}) \cap \mathrm{Hom}(S^2\mathfrak{g}_1, \mathfrak{g}_0)$.

Firstly, we consider a natural basis $\mathcal{B}$ of $\mathrm{Hom}(S^2\mathfrak{g}_1, \mathfrak{g}_0)$ consisting of the following maps where $1 \le s \le n$ and $1 \le i, j, k, l \le m$:

$$\varphi_{i,j}^s(Y_k, Y_l) = \begin{cases} X_s & \text{if } (i, j) = (k, l) \\ 0 & \text{in all other cases.} \end{cases}$$

Thanks to Corollary 4.1.2 it will be enough to find the basis vectors $\varphi^s_{i,j}$ with weight 0 or 1. The weight of an element $\varphi^s_{i,j}$ (with respect to H) is

$$\lambda(\varphi^s_{i,j}) = \lambda(X_s) - \lambda(Y_i) - \lambda(Y_j) = 2m - n + 2(s - i - j) + 1.$$

In fact,

$$\begin{aligned}(H \cdot \varphi^s_{i,j})(Y_i, Y_j) &= H \cdot \varphi^s_{i,j}(Y_i, Y_j) - \varphi^s_{i,j}(H \cdot Y_i, Y_j) - \varphi^s_{i,j}(Y_i, H \cdot Y_j)\\ &= H \cdot X_s - \varphi^s_{i,j}((-m - 1 + 2i)Y_i, Y_j) - \varphi^s_{i,j}(Y_i, (-m - 1 + 2j)Y_j)\\ &= (-n - 1 + 2s)X_s - (-m - 1 + 2i)X_s - (-m - 1 + 2j)X_s\\ &= [2m - n + 2(s - i - j) + 1]X_s.\end{aligned}$$

We are going to introduce a simpler weight of an element $\varphi \in C$. It corresponds to the action of the diagonalizable derivation d, $d \in \mathrm{Der}L^{n,m}$, defined by

$$d(X_0) = X_0, \qquad d(X_i) = iX_i, \qquad d(Y_j) = jY_j; \quad 1 \le i \le n, 1 \le j \le m.$$

This weight will be denoted by $p(\varphi)$. We have that

$$p(\varphi^s_{i,j}) = s - i - j.$$

We have the following relationships between the two weights:

$$\begin{aligned}&\lambda(\varphi) = 2p(\varphi) + 2m - n + 1,\\ &p(\varphi) = \frac{1}{2}(\lambda(\varphi) - 2m + n - 1).\end{aligned}$$

Remark 5.1. If n is even then $\lambda(\varphi)$ is odd, and if n is odd then $\lambda(\varphi)$ is even. So, if n is even it will be sufficient to find the elements $\varphi^s_{i,j}$ with weight 1 and if n is odd it will be sufficient to find those with weight 0.

In order to find the elements with weight 0 or 1, we can consider the three sequences that correspond with the weights of $V_1 = \langle Y_1, Y_2, \ldots, Y_{m-1}, Y_m\rangle$, $V_1 = \langle Y_1, Y_2, \ldots, Y_{m-1}, Y_m\rangle$ and $V_0 = \langle X_1, X_2, \ldots, X_{n-1}, X_n\rangle$:

$$\begin{aligned}&-m + 1, -m + 3, \ldots, m - 3, m - 1;\\ &-m + 1, -m + 3, \ldots, m - 3, m - 1;\\ &-n + 1, -n + 3, \ldots, n - 3, n - 1.\end{aligned}$$

We shall have to count the number of all possibilities to obtain 1 (if n is even) or 0 (if n is odd). Remember that $\lambda(\varphi^s_{i,j}) = \lambda(X_s) - \lambda(Y_i) - \lambda(Y_j)$, where $\lambda(X_s)$ belongs to the last sequence, and $\lambda(Y_i)$, $\lambda(Y_j)$ belong to the first and second sequences respectively. For example, if n is odd we have to obtain 0, so we can fix an element (a weight) of the last sequence and then to count the possibilities to sum the same quantity between the two first sequences. Taking into account the symmetry of $\varphi^s_{i,j}$, that is $\varphi^s_{i,j} = \varphi^s_{j,i}$, and repeating the above reasoning for all the elements of the last sequence we obtain the following theorem:

Theorem 1. *If $C = Z^2(L^{n,m}, L^{n,m}) \cap \mathrm{Hom}(S^2\mathfrak{g}_1, \mathfrak{g}_0)$, then we have the following values for the dimension of C:*

$$\dim C = \begin{cases} \dfrac{m(m+1)}{2} & \text{if } n \ge 2m - 1\\ \dfrac{1}{8}(4mn - n^2 + 2n + 3) & \text{if } n < 2m - 1, n \equiv 1 (\mathrm{mod}\ 4) \text{ and } m \text{ odd, or}\\ & n \equiv 3 (\mathrm{mod}\ 4) \text{ and } m \text{ even}\\ \dfrac{1}{8}(4mn - n^2 + 2n - 1) & \text{if } n < 2m - 1, n \equiv 3 (\mathrm{mod}\ 4) \text{ and } m \text{ odd, or}\\ & n \equiv 1 (\mathrm{mod}\ 4) \text{ and } m \text{ even}\\ \dfrac{1}{8}(4mn - n^2 + 2n) & \text{if } n < 2m - 1 \text{ and } n \text{ even.}\end{cases}$$

Proof. It is convenient to distinguish the following six cases where the reasoning for each case is not hard:

(1) $n \geq 2m - 1$
(2) $n \leq 2m - 1$, n even.
(3) $n \leq 2m - 1$, $n \equiv 1 (\bmod 4)$, m odd.
(4) $n \leq 2m - 1$, $n \equiv 3 (\bmod 4)$, m even.
(5) $n \leq 2m - 1$, $n \equiv 1 (\bmod 4)$, m even.
(6) $n \leq 2m - 1$, $n \equiv 3 (\bmod 4)$, m odd. $\square$

6. Construction of a basis of $Z^2(L^{n,m}, L^{n,m}) \cap \mathrm{Hom}(S^2\mathfrak{g}_1, \mathfrak{g}_0)$

In this section we are going to develop a method that permits us to calculate a basis of $C = Z^2(L^{n,m}, L^{n,m}) \cap \mathrm{Hom}(S^2\mathfrak{g}_1, \mathfrak{g}_0) = C$ in each case.

Let φ be an element of C, with weight $\lambda(\varphi)$. As φ is a maximal vector of the $\mathfrak{sl}(2, \mathbb{C})$-module $\mathrm{Hom}(S^2V_1, V_0)$, its weight $\lambda(\varphi)$ is always a nonnegative integer, $\lambda(\varphi) \geq 0$.

On the other hand, $p(\varphi)$ is always less than or equal to $n-2$, $p(\varphi) \leq n-2$. In fact, $\varphi^n_{1,1}$ is an element with maximal weight $p(\varphi)$, $p(\varphi^n_{1,1}) = n-2$. So, we have the following estimates for $p(\varphi)$:

$$\frac{n-2m-1}{2} \leq p(\varphi) \leq n-2. \tag{6.1}$$

In order to get a basis of C it is enough to obtain the basis for each subspace $C(p)$ of C, spanned by all the elements with weight p such that p satisfies (6.1).
Let $\varphi_{k,s}$ be an element of $\mathrm{Hom}(S^2V_1, V_0)$ with weight p, $p(\varphi_{k,s}) = s - 2k$, and defined by

$$\varphi_{k,s}(Y_i, Y_i) = \begin{cases} X_s & \text{if } i = k \\ 0 & \text{in the other case} \end{cases}$$

with $1 \leq s \leq n$, $1 \leq k \leq m$ and satisfying the equations

$$[X_0, \varphi_{k,s}(Y_i, Y_j)] - \varphi_{k,s}(Y_{i+1}, Y_j) - \varphi_{k,s}(Y_i, Y_{j+1}) = 0, \quad \text{with } 1 \leq i, j \leq m-1. \tag{6.2}$$

Thanks to the Eq. (4.2) we observe that $\varphi_{k,s}$ is not always a cocycle of C. In particular, $\varphi_{k,s}$ will be a cocycle of C if and only if it satisfies the equations

$$[X_0, \varphi_{k,s}(Y_i, Y_m)] - \varphi_{k,s}(Y_{i+1}, Y_m) = 0, \quad \text{with } 1 \leq i \leq m.$$

By induction the following formula for $\varphi_{k,s}$ can be proved:

$$\varphi_{k,s}(Y_i, Y_j) = (-1)^{k-i}\left(C^{k-i}_{j-k} - \frac{1}{2}C^{k-i}_{j-k-1}\right) X_{i+j+s-2k}$$

with $1 \leq i < j \leq m$, $k \leq \frac{i+j}{2}$. We suppose that $C^q_t = 0$ if $q < 0$ or $t < 0$ or $q > t$, and $C^0_0 = C^0_t = 1$ with $t > 0$. In the remaining cases we have $C^q_t = \frac{t!}{q!(t-q)!}$.

Proposition 6.1. *The symmetric bilinear map $\varphi_{k,s}$ defined by the formula*

$$\varphi_{k,s}(Y_i, Y_j) = (-1)^{k-i}\left(C^{k-i}_{j-k} - \frac{1}{2}C^{k-i}_{j-k-1}\right) X_{i+j+s-2k}, \quad 1 \leq i, j \leq m$$

is a cocycle of C iff

$$p(\varphi_{k,s}) = s - 2k \geq n - m - 1.$$

Proof. We only have to check whether $\varphi_{k,s}$ satisfies or does not satisfy the equations

$$[X_0, \varphi_{k,s}(Y_i, Y_m)] = \varphi_{k,s}(Y_{i+1}, Y_m), \quad \text{with } 1 \leq i \leq m.$$

If $p(\varphi_{k,s}) = n - m - 1$, then

$$\varphi_{k,s}(Y_1, Y_m) = (-1)^{k-1}\left(C_{m-k}^{k-1} - \frac{1}{2}C_{m-k-1}^{k-1}\right)X_n$$

and $\varphi_{k,s}(Y_2, Y_m) = \cdots = \varphi_{k,s}(Y_m, Y_m) = 0$ which clearly satisfies the above equations. If $p(\varphi_{k,s}) > n-m-1$, then $\varphi_{k,s}(Y_1, Y_m) = \cdots = \varphi_{k,s}(Y_m, Y_m) = 0$ and this also satisfies the cocycle equations of C.
If $p(\varphi_{k,s}) < n-m-1$, then

$$\varphi_{k,s}(Y_1, Y_m) = (-1)^{k-1}\left(C_{m-k}^{k-1} - \frac{1}{2}C_{m-k-1}^{k-1}\right)X_t$$

with $t < n$. If we apply the cocycle equations we have

$$[X_0, \varphi_{k,s}(Y_1, Y_m)] = \varphi_{k,s}(Y_2, Y_m) = (-1)^{k-2}\left(C_{m-k}^{k-2} - \frac{1}{2}C_{m-k-1}^{k-2}\right)X_{t+1},$$

but

$$\begin{aligned}[X_0, \varphi_{k,s}(Y_1, Y_m)] &= \left[X_0, (-1)^{k-1}\left(C_{m-k}^{k-1} - \frac{1}{2}C_{m-k-1}^{k-1}\right)X_t\right] \\ &= (-1)^{k-1}\left(C_{m-k}^{k-1} - \frac{1}{2}C_{m-k-1}^{k-1}\right), X_{t+1}\end{aligned}$$

and then

$$C_{m-k}^{k-2} - \frac{1}{2}C_{m-k-1}^{k-2} = -C_{m-k}^{k-1} + \frac{1}{2}C_{m-k-1}^{k-1},$$

which is a contradiction. □

Proposition 6.2. *Let $\varphi \in C$ be a cocycle with weight $p = p(\varphi) \le n-m-2$. Then*

$$\varphi = \sum_{s-2k=p} a_k \varphi_{k,s}$$

for some numbers a_k.

Proof. Let $\varphi \in C$ be a cocycle with weight p. Then $\varphi(Y_i, Y_i) = a_i X_{2i+p}$. We are going to consider the difference

$$\Psi = \varphi - \sum_{s-2k=p} a_k \varphi_{k,s}.$$

It is easy to check that Ψ is a symmetric bilinear map such that

$$\Psi(Y_1, Y_1) = \Psi(Y_2, Y_2) = \cdots = \Psi(Y_m, Y_m) = 0.$$

As $\varphi_{k,s}$ satisfies Eq. (6.2), Ψ satisfies it too, which implies that Ψ vanishes. In fact, if we fix i with $1 \le i \le m-1$, we can prove by induction that $\Psi(Y_i, Y_{i+k}) = 0$, for all $k \ge 0$: We suppose that the relation is true up to k. Thanks to (6.2), for $k+1$ we have

$$[X_0, \Psi(Y_i, Y_{i+k})] = 0 = \Psi(Y_{i+1}, Y_{(i+1)+(k-1)}) + \Psi(Y_i, Y_{i+k+1}) = 0 + \Psi(Y_i, Y_{i+k+1})$$

which proves the result. □

Proposition 6.3. *Let φ be a nonzero cocycle of weight $p = p(\varphi) \le n-m-2$*

$$\varphi = \sum_{s-2k=p} a_k \varphi_{k,s}.$$

Then $p \ge n-2m$.

Proof. Let $\varphi = \sum_{s-2k=p} a_k \varphi_{k,s}$ be a cocycle with $p < n-2m$. If φ is nonzero, then there exists i such that $a_i \neq 0$, and thus $\varphi(Y_i, Y_i) = a_i X_{2i+p} \neq 0$. As φ is a cocycle it has to satisfy the Eq. (4.2), and then we have:

$$(ad X_0)(\varphi(Y_i, Y_i)) = 2\varphi(Y_i, Y_{i+1}),$$

$$(adX_0)^2(\varphi(Y_i, Y_i)) = 2\varphi(Y_i, Y_{i+2}) + 2\varphi(Y_{i+1}, Y_{i+1}),$$
$$(adX_0)^3(\varphi(Y_i, Y_i)) = 2\varphi(Y_i, Y_{i+3}) + 6\varphi(Y_{i+1}, Y_{i+2}),$$
$$\vdots$$
$$(adX_0)^{2(m-i)}(\varphi(Y_i, Y_i)) = \sum \alpha_j \varphi(Y_{i+j}, Y_{i+(m-i)+j}) = \alpha\varphi(Y_m, Y_m).$$

Then $0 \neq a_i X_{m+2p} = \alpha\varphi(Y_m, Y_m)$. Thus $\varphi(Y_m, Y_m) = a_m X_{m+2p}$ with $a_m \neq 0$ and $m + 2p < n$, so $[X_0, \varphi(Y_m, Y_m)] = a_m X_{m+2p+1} \neq 0$ which is a contradiction with the Eq. (4.2):

$$[X_0, \varphi(Y_m, Y_m)] = \varphi([X_0, Y_m], Y_m) + \varphi(Y_m, [X_0, Y_m]) = 0. \quad \square$$

Proposition 6.4. *Let φ be a cocycle of C*

$$\varphi = \sum_{s-2k=p} a_k \varphi_{k,s}$$

with $\max\left\{\frac{n-2m-1}{2}, n-2m\right\} \leq p \leq n-m-2$. *Then φ is a cocycle iff*

$$(adX_0)^{r-1}(\varphi(Y_1, Y_m)) = (adX_0)^{r-2}(\varphi(Y_2, Y_m)) = \cdots = (adX_0)(\varphi(Y_{r-1}, Y_m)) = \varphi(Y_r, Y_m)$$

with $r = n - m - p$.

Proof. As each $\varphi_{k,s}$ satisfies Eq. (6.2), φ satisfies it too. Thus, we have that φ will be a cocycle of C iff φ satisfies the equations

$$[X_0, \varphi(Y_i, Y_m)] - \varphi(Y_{i+1}, Y_m) = 0, \quad \text{with } 1 \leq i \leq m$$

which proves the result. $\square$

The above proposition gives us a method for constructing all the cocycles with weight p, $\max\left\{\frac{n-2m-1}{2}, n-2m\right\} \leq p \leq n-m-2$, and combining with Proposition 6.1 the complete description of C can be obtained. An explicit description of C with $n \geq 2m-1$ will be given in the following section.

7. Basis of C for $n \geq 2m - 1$

In this section we are going to apply the method described in the above section in order to construct an explicit basis of C. In particular, we shall give (Theorem 2) a basis of C in the case with more cocycles: $\frac{m(m+1)}{2}$ with $n \geq 2m-1$ (see Theorem 1).

Thanks to the condition $n \geq 2m-1$, the weight p can only be contained in the interval $-1 \leq p \leq n-2$. As Proposition 6.1 gives us the description of the cocycles with $p \geq n-m-1$, it remains to describe a basis of the cocycles of C such that

$$n - 2m \leq p \leq n - m - 2.$$

If we fix p satisfying $n - 2m \leq p \leq n - m - 2$, then all the mappings $\varphi_{k,s}$ with weight p will be

$$\varphi_{1,p+2}, \varphi_{2,p+4}, \ldots, \varphi_{l,p+2l}$$

with $l = \lfloor\frac{n-p}{2}\rfloor$. In fact, as $p \geq n-2m$, then $l = \lfloor\frac{n-p}{2}\rfloor \leq m$ and $\min\left(\lfloor\frac{n-p}{2}\rfloor, m\right) = \lfloor\frac{n-p}{2}\rfloor$. Let φ be

$$\varphi = a_1\varphi_{1,p+2} + a_2\varphi_{2,p+4} + \cdots + a_l\varphi_{l,p+2l}.$$

To simplify the expressions, we shall denote by $\overline{C}_j^k$ the expression $C_j^k - \frac{1}{2}C_{j-1}^k$.

Proposition 6.4 gives us $r - 1 = n - m - p - 1$ linear equations in $a_1, \ldots, a_l$:

$$(adX_0)^i(\varphi(Y_{r-i}, Y_m)) = \varphi(Y_r, Y_m), \quad 1 \leq i \leq r-1.$$

That is, if $p > n-2m$ the resulting system is

$$\begin{aligned}
&\frac{1}{2}a_1 - a_2\overline{C}^1_{m-2} + a_3\overline{C}^2_{m-3} + \cdots + a_l(-1)^{l-1}\overline{C}^{l-1}_{m-l} = \frac{1}{2}a_r + \cdots + a_l(-1)^{l-r}\overline{C}^{l-r}_{m-l}\\
&\frac{1}{2}a_2 - a_3\overline{C}^1_{m-3} + \cdots + a_l(-1)^{l-2}\overline{C}^{l-2}_{m-l} = \frac{1}{2}a_r + \cdots + a_l(-1)^{l-r}\overline{C}^{l-r}_{m-l}\\
&\vdots\\
&\frac{1}{2}a_{r-1} + \cdots + a_l(-1)^{l-r+1}\overline{C}^{l-r+1}_{m-l} = \frac{1}{2}a_r + \cdots + a_l(-1)^{l-r}\overline{C}^{l-r}_{m-l},
\end{aligned}$$

and if $p = n-2m$, then $r = m$, and thus the coefficient of $a_r = a_m$ will be 1 instead of $\frac{1}{2}$. Then the system is

$$\begin{aligned}
&\frac{1}{2}a_1 - a_2\overline{C}^1_{m-2} + a_3\overline{C}^2_{m-3} + \cdots + a_{m-1}(-1)^{m-2}\overline{C}^{m-2}_1 = 1\\
&\frac{1}{2}a_2 - a_3\overline{C}^1_{m-3} + \cdots + a_{m-1}(-1)^{m-3}\overline{C}^{m-3}_1 = 1\\
&\vdots\\
&\frac{1}{2}a_{m-1} = 1.
\end{aligned}$$

The basis of the set of solutions of this last system can be obtained by induction in the following way: $a_{m-1} = 2$; $a_{m-2} = 2 + 2\overline{C}^1_1(a_{m-1})$; $a_{m-3} = 2 + 2\overline{C}^1_2(a_{m-2}) - 2\overline{C}^2_1(a_{m-1})\ldots$, that is

$$a_i = 2 + 2\sum_{j=1}^{m-i-1}(-1)^{j+1}\overline{C}^j_{m-i-j}(a_{i+j}), \quad i = m-1, m-2, \ldots, 1.$$

The recursion formula is easy to apply for specific values of m and n. Developing the recursion formula we obtain an explicit expression for a_i:

$$a_i = 2 + \sum_{q=2}^{m-i} 2^q \left[\prod_{k=1}^{q-1} \sum_{j_k=1}^{m-i-1-\sum_{t=1}^{k-1} j_t} (-1)^{j_k+1}\overline{C}^{j_k}_{m-i-\sum_{t=1}^{k} j_t}\right], \quad 1 \le i \le m-1.$$

For these values of a_i we obtain the following cocycle that corresponds to the value 1 of the coefficient of $\varphi_{m,n}$. Thus, we shall call it $\overline{\varphi}_{m,n}$:

$$\overline{\varphi}_{m,n} = a_1\varphi_{1,n-2m+2} + a_2\varphi_{2,n-2m+4} + \cdots + a_{m-1}\varphi_{m-1,n-2} + \varphi_{m,n}.$$

In the case $p > n-2m$ the system admits $l - r + 1 = \lfloor\frac{n-p}{2}\rfloor - n + m + p + 1$ linearly independent solutions that correspond to the following possibilities for the vector $(a_r, a_{r+1}, \ldots, a_l)$:

$$(1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1).$$

- If $(a_r, a_{r+1}, \ldots, a_l) = (1, 0, \ldots, 0)$, the resulting system is given by

$$\begin{aligned}
&\frac{1}{2}a_1 - a_2\overline{C}^1_{m-2} + a_3\overline{C}^2_{m-3} + \cdots + a_{r-1}(-1)^{r-2}\overline{C}^{r-2}_{m-r+1} = \frac{1}{2} + (-1)^r\overline{C}^{r-1}_{m-r}\\
&\frac{1}{2}a_2 - a_3\overline{C}^1_{m-3} + \cdots + a_{r-1}(-1)^{r-3}\overline{C}^{r-3}_{m-r+1} = \frac{1}{2} + (-1)^{r-1}\overline{C}^{r-2}_{m-r}\\
&\vdots\\
&\frac{1}{2}a_{r-1} = \frac{1}{2} + \overline{C}^1_{m-r}
\end{aligned}$$

whose solution basis can be obtained by induction in the following way:

$a_{r-1} = 1 + 2\overline{C}^1_{m-r}$; $a_{r-2} = 1 - 2\overline{C}^2_{m-r} + 2\overline{C}^1_{m-r+1}(a_{r-1})\ldots$, that is

$$a_i = 1 + 2(-1)^{r-i+1}\overline{C}^{r-i}_{m-r} + 2\sum_{j=1}^{r-i-1}(-1)^{j+1}\overline{C}^{j}_{m-i-j}(a_{i+j}), \quad i = r-1, r-2, \ldots, 1.$$

Developing the recursion formula we obtain an explicit expression for a_i:

$$\begin{aligned} a_i &= 1 + 2(-1)^{r-i+1}\overline{C}^{r-i}_{m-r} + \sum_{q=2}^{r-i}2^{q-1}\left[\prod_{k=1}^{q-1}\sum_{j_k=1}^{r-i-1-\sum_{t=1}^{k-1}j_t}(-1)^{j_k+1}\overline{C}^{j_k}_{m-i-\sum_{t=1}^{k}j_t}\right] \\ &\quad + \sum_{q=2}^{r-i}2^q(-1)^{r-i+q}\left(\prod_{k=1}^{q-1}\sum_{j_k=1}^{r-i-1-\sum_{t=1}^{k-1}j_t}\overline{C}^{j_k}_{m-i-\sum_{t=1}^{k}j_t}\right)\left(\overline{C}_{m-r}^{r-i-\sum_{t=1}^{q-1}j_t}\right) \end{aligned}$$

with $1 \le i \le r-1$. Hence for these values of a_i $(1 \le i \le r-1)$ and $a_r = 1, a_{r+1} = \cdots = a_l = 0$, we obtain the cocycle $\overline{\varphi}_{r,p+2r}$ given by

$$\overline{\varphi}_{r,p+2r} = a_1\varphi_{1,p+2} + a_2\varphi_{2,p+4} + \cdots + a_{r-1}\varphi_{r-1,p+2(r-1)} + \varphi_{r,p+2r}.$$

- In the remaining cases, it can be seen that for each h, $r+1 \le h \le l$, such that $a_h = 1$ and $a_k = 0$ with $r \le k \le l$ and $k \ne h$, we obtain the cocycle $\overline{\varphi}_{h,p+2h}$,

$$\overline{\varphi}_{h,p+2h} = a^h_1\varphi_{1,p+2} + a^h_2\varphi_{2,p+4} + \cdots + a^h_{r-1}\varphi_{r-1,p+2(r-1)} + \varphi_{h,p+2h}$$

with

$$\begin{aligned} a^h_i &= 2(-1)^{h-r}\overline{C}^{h-r}_{m-h} + (-1)^{h-r}\overline{C}^{h-r}_{m-h}\sum_{q=2}^{r-i}2^q\left[\prod_{k=1}^{q-1}\sum_{j_k=1}^{r-i-1-\sum_{t=1}^{k-1}j_t}(-1)^{j_k+1}\overline{C}^{j_k}_{m-i-\sum_{t=1}^{k}j_t}\right] \\ &\quad + 2(-1)^{h-i+1}\overline{C}^{h-i}_{m-h} + \sum_{q=2}^{r-i}2^q(-1)^{h-i+q}\left(\prod_{k=1}^{q-1}\sum_{j_k=1}^{r-i-1-\sum_{t=1}^{k-1}j_t}\overline{C}^{j_k}_{m-i-\sum_{t=1}^{k}j_t}\right)\left(\overline{C}_{m-h}^{h-i-\sum_{t=1}^{q-1}j_t}\right) \end{aligned}$$

for $1 \le i \le r-1$.

Thus, we have the following

Theorem 2. *A basis of the space of cocycles $Z^2(L^{n,m}, L^{n,m}) \cap \mathrm{Hom}(S^2\mathfrak{g}_1, \mathfrak{g}_0)$, with $n \ge 2m-1$, will be given by the following cocycles.*

- *For each p such that $n-m-1 \le p \le n-2$, there are $\lfloor\frac{n-p}{2}\rfloor$ cocycles of weight p in the basis, that is*

$$\varphi_{1,p+2}, \varphi_{2,p+4}, \ldots, \varphi_{\lfloor\frac{n-p}{2}\rfloor, p+2\lfloor\frac{n-p}{2}\rfloor}.$$

- *For $p = n-2m$, there is only one cocycle of weight p in the basis, that is*

$$\overline{\varphi}_{m,n} = a_1\varphi_{1,n-2m+2} + a_2\varphi_{2,n-2m+4} + \cdots + a_{m-1}\varphi_{m-1,n-2} + \varphi_{m,n}$$

with

$$a_i = 2 + \sum_{q=2}^{m-i}2^q\left[\prod_{k=1}^{q-1}\sum_{j_k=1}^{m-i-1-\sum_{t=1}^{k-1}j_t}(-1)^{j_k+1}\overline{C}^{j_k}_{m-i-\sum_{t=1}^{k}j_t}\right], \quad 1 \le i \le m-1.$$

- *For each p such that $n-2m < p \leq n-m-2$, there are $l-r+1 = \lfloor \frac{n-p}{2} \rfloor - (n-m-p)+1$ cocycles of weight p in the basis, that is*

$$\overline{\varphi}_{r,p+2r} = a_1\varphi_{1,p+2} + a_2\varphi_{2,p+4} + \cdots + a_{r-1}\varphi_{r-1,p+2(r-1)} + \varphi_{r,p+2r}$$

with

$$a_i = 1 + 2(-1)^{r-i+1}\overline{C}_{m-r}^{r-i} + \sum_{q=2}^{r-i} 2^{q-1} P_q + \sum_{q=2}^{r-i} 2^q (-1)^{r-i+q} R_q \left(\overline{C}_{m-r}^{\,r-i-\sum_{t=1}^{q-1} j_t} \right),$$

and

$$\overline{\varphi}_{h,p+2h} = a_1^h\varphi_{1,p+2} + a_2^h\varphi_{2,p+4} + \cdots + a_{r-1}^h\varphi_{r-1,p+2(r-1)} + \varphi_{h,p+2h}$$

with

$$a_i^h = 2(-1)^{h-r}\overline{C}_{m-h}^{h-r} + (-1)^{h-r}\overline{C}_{m-h}^{h-r}\sum_{q=2}^{r-i} 2^q P_q + 2(-1)^{h-i+1}\overline{C}_{m-h}^{h-i}$$
$$+ \sum_{q=2}^{r-i} 2^q (-1)^{h-i+q} R_q \left(\overline{C}_{m-h}^{\,h-i-\sum_{t=1}^{q-1} j_t} \right)$$

for $r < h \leq l$, $1 \leq i \leq r-1$. Here we have denoted by P_q and R_q the following sequences of nested products:

$$P_q = \prod_{k=1}^{q-1} \sum_{j_k=1}^{r-i-1-\sum_{t=1}^{k-1} j_t} (-1)^{j_k+1}\overline{C}^{j_k}_{m-i-\sum_{t=1}^{k} j_t}$$

$$R_q = \prod_{k=1}^{q-1} \sum_{j_k=1}^{r-i-1-\sum_{t=1}^{k-1} j_t} \overline{C}^{j_k}_{m-i-\sum_{t=1}^{k} j_t}.$$

Proof. For each p all the cocycles described in the theorem are linearly independent, so all the cocycles of the theorem are linearly independent. It remains to count them.

In the case $n-m-1 \leq p \leq n-2$ the cocycles obtained are all of the form $\varphi_{k,s}$ (see Proposition 6.1). In particular there are $\lfloor \frac{n-p}{2} \rfloor$ cocycles for each p; thus in total for this case we have

$$\sum_{p=n-m-1}^{n-2} \left\lfloor \frac{n-p}{2} \right\rfloor = \begin{cases} \dfrac{m^2+2m}{4} & \text{if } m \text{ is even} \\ \dfrac{m^2+2m+1}{4} & \text{if } m \text{ is odd.} \end{cases}$$

In the case $n-2m \leq p \leq n-m-2$, we have

$$\sum_{p=n-2m}^{n-m-2} \left\lfloor \frac{n-p}{2} \right\rfloor - n + m + p + 1 = \begin{cases} \dfrac{m^2}{4} & \text{if } m \text{ is even} \\ \dfrac{m^2-1}{4} & \text{if } m \text{ is odd.} \end{cases}$$

If we sum, we obtain in total $\frac{m(m+1)}{2}$ cocycles of $Z^2(L^{n,m}, L^{n,m}) \cap \mathrm{Hom}(S^2\mathfrak{g}_1, \mathfrak{g}_0)$ that are linearly independent. As the dimension of the space is $\frac{m(m+1)}{2}$ (see Theorem 1) the proof is finished. □

Acknowledgements

This work was partially supported by Junta de Extremadura-Consejería de Infraestructuras y Desarrollo Tecnológico and by Feder (N. 3PR05A074).

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